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# An Extension to *BMO* Functions of Some Product Properties of Hilbert Transforms

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Let $b \in BMO$, $f \in L^p$, $1 < p < \infty$, $H'b$ denote the dual Hilbert transform of $b$ defined on $BMO$ according to the $H^1$–$BMO$ duality and $H$ be the Hilbert transform defined as the pointwise limit almost everywhere of appropriate truncated integrals. We prove that $H'b.Hf - bf = H(bHf) + H(fH'b)$ a.e. The proof uses distributional convolutions with $vp(1/x)$ and a result of R. R. Coifman, R. Rochberg, and G. Weiss on the $L^p$-boundedness of some commutators. © 1987 Academic Press, Inc.

## 1. NOTATION

We consider real valued functions. The symbol $H$ denotes the Hilbert transform defined as the following pointwise limit a.e.:

$$HF(x) = \lim_{\varepsilon \to 0, r \to \infty} \int_{0 < \varepsilon < |x-t| < r} \frac{F(t)}{\pi(x-t)}\, dt.$$

The dual Hilbert transform on $BMO$ is denoted by $H'$; thus,

$$\langle H'b, h \rangle = -\langle b, Hh \rangle \qquad \forall b \in BMO, \qquad \forall h \in H^1.$$

As in [4], the space $(1+x^2)^{1/2}\, \mathscr{D}'_{L^1} = \mathscr{D}'_{L^1} + x\mathscr{D}'_{L^1}$ is the set of all distributions $T$ such that $(1+x^2)^{-1/2}\, T \in \mathscr{D}'_{L^1}$ or, what is equivalent, the set of all distributions of the form $T = T_1 + xT_2$ with $T_1 \in \mathscr{D}'_{L^1}$, $T_2 \in \mathscr{D}'_{L^1}$. The topology on this space is defined according to [4] also, i.e., $T$ is said to converge to zero in $(1+x^2)^{1/2}\, \mathscr{D}'_{L^1}$ if $(1+x^2)^{-1/2}\, T$ converges to zero in $\mathscr{D}'_{L^1}$.

For $T \in \mathscr{D}'_{L^1} + x\mathscr{D}'_{L^1}$, we also use the convolution $T * vp(1/\pi x)$ as the latter is defined in [4], i.e., for every $\varphi \in \mathscr{S}$,

$$\left\langle T * vp\frac{1}{\pi x}, \varphi \right\rangle = -\left\langle (1+x^2)^{-1/2}\, T, (1+x^2)^{1/2}\left(vp\frac{1}{\pi x} * \varphi\right)\right\rangle$$

The operator $\mathscr{H}$: $T \mapsto T * vp(1/\pi x)$ is continuous from $\mathscr{D}'_{L^1} + x\mathscr{D}'_{L^1}$ into $\mathscr{S}'$



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and coincides with the usual Hilbert transform on $L^p$, $1<p<\infty$ [4]. Moreover, it coincides with the Hilbert transform on $H^1$ too, since, for every $\varphi \in \mathscr{S}$ and $f \in H^1$,

$$_{\mathscr{S}'}\langle \mathscr{H}f, \varphi \rangle_{\mathscr{S}} = {}_{\mathscr{S}'}\left\langle vp \frac{1}{\pi x} * f, \varphi \right\rangle_{\mathscr{S}} = -{}_{L^1}\langle f, H\varphi \rangle_{L^\infty}$$

$$= (f * H\check{\varphi})(o) = (Hf * \check{\varphi})(o) = \langle Hf, \varphi \rangle,$$

where $\check{\varphi}(t) = \varphi(-t)$ and the third equality results from [1], Proposition 8.2.3.

It is known that $BMO$ functions do not necessarily belong to $\mathscr{D}'_{L^1} + x\mathscr{D}'_{L^1}$ (for example, sign $x$ does not). This justifies the choice of a distinct notation for the dual Hilbert transform $H'$ on $BMO$ and the above distributional one.

Besides, the multiplication operator by $f$ is denoted by $M(f)$ and the commutator of $A$, $B$ by $[A, B] = AB - BA$. The following abbreviations are used also:

$wL^1$ instead of $(1+x^2)^{1/2} L^1 = L^1((1+x^2)^{-1/2}\, dx)$,

$w^{-1}\mathscr{B}$ to denote $\{g \in C^\infty : \sup_{\mathbb{R}}[(1+x^2)^{1/2}\, |D^K g(x)|] < \infty, \forall K \in \mathbb{N}\}$,

$L^p_c$ to denote the set of all $L^p$-functions with compact support.

## 2. Product Properties

2.1. Theorem. *If $b \in BMO$, $f \in L^p$, $1<p<\infty$, then the following identity holds almost everywhere*:

$$H'b.Hf - bf = H(b.Hf) + H(f.H'b). \tag{1}$$

2.2. *Remarks.* (i) In (1), $b$ and $H'b$ are defined modulo constants. When $b$ and $H'b$ are, respectively, replaced by $b + c_1$, $H'b + c_2$, where $c_1$, $c_2$ are constants, the same function $c_2 Hf - c_1 f$ is added to both members of (1). Thus, identity (1) holds a.e. whatever the choice of the constants $c_1$, $c_2$ may be.

(ii) As a corollary of (1), we obtain the inverse identity $b.Hf + f.H'b = -H(H'b.Hf - fb)$ a.e., by replacing $b$ by $H'b$ or $f$ by $Hf$.

*Proof of Theorem* 2.1. We first suppose that $\varphi \in \mathscr{S}$ or $\varphi \in L^p_c$, for some $1<p<\infty$, and $b \in BMO$. By [2], we have

$$\varphi.H\Psi + \Psi.H\varphi = -H(H\varphi.H\Psi - \varphi\Psi), \qquad \forall \Psi \in \mathscr{S}.$$

Thus,

$$
\begin{aligned}
&{}_{BMO}\langle b, \varphi.H\Psi + \Psi.H\varphi\rangle_{H^1}\\
&\quad = -{}_{BMO}\langle b, H(H\varphi \cdot H\Psi - \varphi\Psi)\rangle_{H^1}\\
&\quad = {}_{BMO}\langle H'b, H\varphi.H\Psi - \varphi\Psi\rangle_{H^1}\\
&\quad \overset{(\times)}{=} {}_{wL^1}\langle H\varphi.H'b, H\Psi\rangle_{w^{-1}\mathscr{B}} - {}_{\mathscr{S}'}\langle \varphi.H'b, \Psi\rangle_{\mathscr{S}}\\
&\quad = -{}_{\mathscr{S}'}\langle \mathscr{H}(H\varphi.H'b) + \varphi.H'b, \Psi\rangle_{\mathscr{S}}, \qquad \forall \Psi \in \mathscr{S},
\end{aligned}
$$

where $(\times)$ is justified by the fact that $|H\varphi(x)| \leqslant c\,|x|^{-1}$ as $|x| \to \infty$ when $\varphi \in \mathscr{S}$ or $L^p_c$.

On the other hand,

$$
\begin{aligned}
&{}_{BMO}\langle b, \varphi.H\Psi + \Psi.H\varphi\rangle_{H^1} = {}_{wL^1}\langle \varphi b, H\Psi\rangle_{w^{-1}\mathscr{B}} + {}_{\mathscr{S}'}\langle b.H\varphi, \Psi\rangle_{\mathscr{S}}\\
&= {}_{\mathscr{S}'}\langle -\mathscr{H}(\varphi b) + b.H\varphi, \Psi\rangle_{\mathscr{S}}, \qquad \forall \Psi \in \mathscr{S}.
\end{aligned} \tag{2}
$$

Therefore,

$$\varphi.H'b + b.H\varphi = -\mathscr{H}(H\varphi.H'b) + \mathscr{H}(\varphi b) \tag{3}$$

holds in $\mathscr{S}'$, or, which is equivalent,

$$[M(b), \mathscr{H}]\,\varphi = [M(H'b), \mathscr{H}](H\varphi) \tag{4}$$

in $\mathscr{S}'$.

Now, for $1/p + 1/q = 1$, we have by [2],

$$
\begin{aligned}
\|\varphi.H\Psi + \Psi.H\varphi\|_{H^1} &= \|\varphi.H\Psi + \Psi.H\varphi\|_{L^1} + \|H\varphi.H\Psi - \varphi\Psi\|_{L^1}\\
&\leqslant c\,\|\varphi\|_{L^p}\,\|\Psi\|_{L^q}.
\end{aligned}
$$

Thus, it results from (2) that $[M(b), \mathscr{H}]\,\varphi \in L^p$. Moreover, both commutators in (4) extend on $L^p$. For every $f \in L^p$, $1 < p < \infty$, we have

$$\overline{[M(b), \mathscr{H}]}f = \overline{[M(H'b), \mathscr{H}]}(Hf) \text{ in } L^p \text{ and a.e.,} \tag{5}$$

and

$$\|\overline{[M(b), \mathscr{H}]}\|_{p,p} = \|\overline{[M(H'b), \mathscr{H}]}\|_{p,p} \leqslant c\,\|b\|_{BMO},$$

where $\overline{[.]}$ denotes the closure of the operator.

Besides, $[M(b), \mathscr{H}]$ and $[M(b), H]$ coincide on $L^p_c$, $1 < p < \infty$. Indeed, $b \in L^r_{\mathrm{loc}}$, for every $1 < r < \infty$, and thus $b\varphi \in L^s$ when $1/r + 1/p = 1/s < 1$ which implies $H(b\varphi) = \mathscr{H}(b\varphi)$ a.e. Furthermore, by [3], we know that

$[M(b), H]$ is bounded from $L^p$ into $L^p$, $1<p<\infty$. Consequently, (5) can be written as $[M(b), H]f=[M(H'b), H](Hf)$ a.e., $\forall f\in L^p$, $1<p<\infty$, which is an alternative formulation of the thesis.

*Further Remarks.* (1) Both members of (3) belong to $(1+x^2)^{1/2} L^1$. By replacing $b$ by $H'b$, we also have, for $b\in BMO$, $\varphi\in\mathscr{S}$ or $L^p_c$, $1<p<\infty$,

$$H'b.H\varphi - b\varphi = \mathscr{H}(\varphi H'b + bH\varphi),$$

which proves that $\mathscr{H}^2=-I$ on the subspace

$$\{bH\varphi+\varphi.H'b\colon b\in BMO,\ \varphi\in\mathscr{S} \text{ or } L^p_c,\ 1<p<\infty\} \text{ of } (1+x^2)^{1/2} L^1.$$

(2) The following pointwise approximations can be deduced from the proof of theorem 2.1:

$$H(fb)=\lim \mathscr{H}(\varphi_{k_j} b) \text{ a.e.},$$

$$H(Hf.H'b)=\lim \mathscr{H}(H\varphi_{k_j}.H'b) \text{ a.e.},$$

for every $b\in BMO$, $f\in L^p$, $1<p<\infty$, where $\varphi_{k_j}$ is a suitable subsequence of $\varphi_k\in L^p_c$, $\varphi_k$ tending to $f$ in $L^p$.

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